Convergence of the sequence kn-p and its properties

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Abstract

Define a sequence $\mathcal{S}_n^k = \{a_1, a_2, a_3 \dots\}$ by setting that $a_1 = n$ and $a_{i+1} = ka_i - p_i$ where p_i is the biggest prime number less than ka_i. There are k's such that $\inf \mathcal{S}_n^k = 1$ for any $n \in \mathbb{N}$, and we will say that such k's have the property all-to-one. Briefly, if a natural number k has all-to-one property, we will say that k is all-to-one. In this paper, we will provide the list of k's with all-to-one property under 28314000 using Nagura and Ramare's result and an algorithm to find all-to-one k. Also, we show that if $1 < \inf \mathcal{S}_n^k$ for some k, there must be a repeated subsequence of numbers, an orbit in $\mathcal{S}_n^k$, moreover it is an equivalent condition for k not to be all-to-one. Furthermore, we will show that for any all-to-one k we can represent any natural number in the form of power series of k^{-1} with prime coefficients. Finally we expand this sequence using more general functional operations on natural numbers, i.e. define the sequence $\mathcal{S}_n^f$ by $a_{i+1} = f(a_i) - p_i$ for some function $f : \mathbb{N} \to \mathbb{N}$. Using similar notation, we say that a function f is all-to-one if $\inf \mathcal{S}_n^f$ for every natural number n. We will provide few functions with all-to-one property and suggest a conjecture on a function to be all-to-one.

1. Introduction

Prime numbers have been one of the biggest concerns in the field of number theory since Euclid, especially the distribution of prime numbers was the most important and difficult conundrum of the 19th and 20th century, even until now. While Riemann's hypothesis suggests the distribution law of prime numbers very precisely, there are some 'naïve' theorems and conjectures on distributions. 'Bertrand's postulate' is one of the earlist approach on the bound of prime counting function $\pi(x)$. [1]

The postulate states that there always exists at least one prime number between x and $2x$ for any $x > 1$. The very first complete proof of this conjecture was provided by Cheybshev in 1852. After that, Ramanujan's brief proof [2] and Erdös's 'elementary' proof followed [3]. Since then, many improvements have been made by lots of mathematicians. From the fact that the prime number theorem implies that the first Chebyshev function $\vartheta(x) \sim x$, one can think of a generalised version of Bertrand's postulate, which must be in the form of 'there exists at least one prime number in $[x, Ax]$ for some $A \in \mathbb{R}_+$ for all $x \geq B$ for some $B \in \mathbb{R}_+$'. Jitsuro Nagura gave the first generalised Bertrand's postulate - a better linear bound for prime numbers - in 1952 [4], and many better results on the problem has followed so far.

In this paper, we will study on '$kn - p$' sequence, which is related to these bounds and suggest some conjectures. First, let us define $kn - p$ sequence. For any natural number n and some k, find the biggest prime number p (strictly) less than kn. We can obtain a new natural number $kn - p$ by subtracting p from kn. Now we can get a sequence $n, kn - p, \dots$ by repeating the same action we did for n on $kn - p$. We want to verify whether it converges, and if so, which number it will end up with. Interestingly, when $k = 4$, the sequence always converges to 1 regardless which number you start with, big or small. For example, let us start with 1000. The biggest prime smaller than 4000 is 3989 and thus we have $4000 - 3989 = 11$ now. The biggest prime number smaller than 44 is 43, hence we have $44 - 43 = 1$.

Now we wonder, would $3n - p$ always converge to 1? Or possibly $2412n - p$? The answer is yes. But it does not hold for $5n - p$ (when n = 2) or $712n - p$ (when n = 13). We will provide an algorithm to verify if $kn - p$ converges to 1 for any n for some k, and provide a list of such k's under 28314000 using the algorithm. Before we start, since the notation $kn - p$ does not seem so very mathematical, we will use these definitions in this paper to state this more elegantly.

Definition 1. For $k, n \in \mathbb{N}$, $\mathcal{S}_n^k = \{a_1, a_2, a_3 \dots\}$ is a sequence such that, $a_1 = n$ and $a_{i+1} = ka_i - p_i$ where p_i is the biggest prime number less than ka_i. If such p_i does not exist, $a_{i+1} = a_i$.

Definition 2. If $k \in \mathbb{N}$ satisfies $\inf \mathcal{S}_n^k = 1$ for all $n \in \mathbb{N}$, we say that k has all-to-one property. Equivalently, k is **all-to-one**. If k is not all-to-one, we say k is **not-all-to-one**.

Observation 1. Using Dirichlet's theorem, [5] one can easily show that there should be n such that $kn - 1$ is a prime and hence $\inf_{n \in \mathbb{N}} \mathcal{S}_n^k = 1$ for any k and there cannot be all-to-j k with $j \neq 1$.

Our first goal in this paper will be establishing an algorithm to determine whether k is all-to-one or not, and secondly to find out some properties of not-all-to-one k's, finally show that any $n \in \mathbb{N}$ can be represented in a power series of k^{-1} where k is all-to-one with prime coefficients. Before we discuss the main theme, let us define terms to use in this paper.

Definition 3. If a subsequence of numbers appear repeatedly in the sequence $\mathcal{S}_n^k$, we call it an **orbit**. We do not consider orbits containing 1.

Definition 4. We call the smallest element s of $\mathcal{S}_n^k$ a **start of the orbit**.

In the following sections, we provide the properties an orbit can have corresponding the parity of k, and the statistics on orbits.

2. all-to-one k

2.1. Existence of all-to-one k

Theorem 1. *There exists at least 4 all-to-one k's and their values are 2, 3, 4, 6....*

Proof. In 1952, Jitsuro Nagura proved that [4]

$$\textit{There exists at least one prime number } p \in [x, \frac{6}{5}x] \text{ for all } x \geq 25$$

Using this result, for any n such that $\frac{5}{6}kn \geq 25$, we have there must be a prime p where $\frac{5}{6}kn < p < kn$. This suggests that $\mathcal{S}_n^k(2) = kn - p_1 < kn - \frac{5}{6}kn = \frac{1}{6}kn < n$. This yields that for any $n \in \mathbb{N}$, $\inf \mathcal{S}_n^k < \frac{30}{k}$. Therefore we only have to check the cases $n < \frac{30}{k}$ to verify where k is all-to-one or not. By direct calculations, we have $k = 2, 3, 4, 6$ is all-to-one. □

Like this, when a linear bound with at least one prime is given, we can verify finite natural number k's to be all-to-one or not. If the number gets bigger, we can use prime gaps to reduce the essential cases to verify whether k is all-to-one.

Theorem 2. *There are 5004113 all-to-one k's under 28314000 and the orbits for not-all-to-one k's under 28314000 does not have any components exceed 382.*

Proof. In 2003, Olivier Ramare [6] suggested a better linear bound on prime, such that

$$\textit{There exists at least one prime number } p \in \left[x\left(1 - \frac{1}{28314000}\right), x\right] \textit{ for all } x \geq 10726905041$$

For $k \leq 28314000$ and $n \in \mathbb{N}$ such that $kn \leq 10726905041$, using similar analogy to 2.1, we have $\mathcal{S}_n^k(2) = kn - p_1 < n$. It implies that $k{\cdot}\inf \mathcal{S}_n^k < 10726905041$. Note that the largest prime gap before 10726905041 is 382 from Young and Potler's result [7], we can easily deduce that $\inf \mathcal{S}_n^k < 382$. Hence we only have to check the cases with $n \leq 382$ to determine whether k is all-to-one or not. Furthermore, we can deduce that every orbits for $k \leq 28314000$ does not have any components exceed 382. □

From these observations, we can deduce that if a nonlinear bound (which is better than linear bounds) for a prime number is given, we can verify any k to be all-to-one. Here we provide an example, while one can find their own verifying method using any given upper bounds and lower bounds of the prime counting function or Chebyshev functions, we use Pierre Dusart's bounds on first Chebyshev function [8]. Since it is known that the first Chebyshev function is almost linear, it is much easier to find a proper upper bound for the numbers we have to check.

Theorem 3. *For any k, we can verify whether it is all-to-one or not with finite calculations. Moreover, we only have to check numbers under the largest prime gap before*

$$max\left(\frac{10544111k}{k-1}, e^{(2k-1)\alpha}\right),\ \alpha = 0.006788$$

Proof. Let us use similar analogy. For any k, we want $\exists$ prime p such that $kn - n \leq p \leq kn$. Then regarding the first Chebyshev function $\vartheta(x) = \sum_{p \leq x} \log p$, we want $\vartheta(kn) - \vartheta((k-1)n) > 0$. Introducing the upper and lower bound of Chebyshev function calculated by Pierre Dusart, $|\vartheta(x) - x| \leq 0.006788\frac{x}{\ln x}$ for $x \geq 10544111$,

$$\begin{aligned}\vartheta(kn) - \vartheta((k-1)n) &\geq n - 0.006788\left(\frac{kn}{\log kn} + \frac{(k-1)n}{\log (k-1)n}\right) \\ &\geq n - 0.006788\left(\frac{(2k-1)n}{\log kn}\right) = n\left(1 - 0.006788\frac{2k-1}{\log kn}\right)\end{aligned}$$

To have this strictly bigger than 0, we want $1 - 0.006788\frac{2k-1}{\log kn} > 0$, i.e. $n > \frac{1}{k}e^{(2k-1)\alpha}$ ($\alpha = 0.006788$). Since $(k-1)n > 10544111$ to have upper bound of $\vartheta((k-1)n)$ right, we conclude that

$$\textit{If } n > \max\left(\frac{10544111}{k-1}, \frac{1}{k}e^{(2k-1)\alpha}\right),\ kn - p < n \qquad p \textit{ is a prime, } \alpha = 0.006788$$

Since $kn - p$ must be equal or smaller than the largest prime gap g before kn, we only have to check numbers under g. □

But as an exponential function grows very fast, it can be challenging to check if a big number k is all-to-one or not. If a sharper bound for prime counting functions is given, we will be able to verify more k's.

Corollary 4. $\sup \mathcal{S}_n^k < \infty$ *for any* k, n

Proof. Since we have shown that the sequence decreases in the very first and then there only appears numbers smaller than some prime gap g, supremum of the sequence $\mathcal{S}_n^k$ should be finite.

Corollary 5. *The number* k *is not-all-to-one, if and only if* $\exists n \in \mathbb{N}$ *where* $\mathcal{S}_n^k$ *contains an orbit*

Proof. Since $\inf \mathcal{S}_n^k < g$ for some g, if k is not-all-to-one, there must be an orbit by the pigeonhole principle. The other way round is trivial$\mathcal{S}_n^k$.

From this corollary, we can learn the fact that we only need to check if there is any orbit to verify if it is all-to-one k or not. In other words, the sequence $\mathcal{S}_n^k$ never diverges.

2.2. Algorithm to reduce the calculations

We have shown that it is possible to check if a k is all-to-one or not in the previous section. But still it needs too many calculations to check whether $\inf \mathcal{S}_n^k = 1$. To avoid the problem, we will establish a better algorithm for the verification in this section.

Observation 2. $\mathcal{S}_n^k = \{n, \mathcal{S}_n^k(2), \mathcal{S}_n^k(3), \dots\} = \{n\} \cup \mathcal{S}_k^{\mathcal{S}_n^k(2)} = \{n, \mathcal{S}_n^k(2)\} \cup \mathcal{S}_k^{\mathcal{S}_n^k(3)} = \dots$

Obviously, if we start sequence from the second element of the sequence, you will get the exactly same subsequence from the second element. That is, when verifying, we do not need to proceed verifying if we get a number we have already checked.

Lemma 6. $(k, \mathcal{S}_n^k(i)) = (\mathcal{S}_n^k(i), \mathcal{S}_n^k(i-1)) = 1$ *for* $i \geq 2$, $\forall k, n \in \mathbb{N}$

Proof. Let $m = \mathcal{S}_n^k(i-1)$. Let p be the largest prime number less than km, then $km - p = \mathcal{S}_n^k(i)$. Now, $d = (km, km-p) = (km, p)$ is either 1 or p. Suppose $d = p$, then $p \leq km - p$ implies that $km \geq 2p$. But Bertrand postulate implies that $\exists$ prime q such that $p < q < 2p \leq km$, which is contradiction to the maximality of p. Hence $(km, km-p) = 1$ and it implies that $(k, km-p) = (m, km-p) = 1$.

Corollary 7. *Any number* a *where* $gcd(a,k) > 1$ *cannot appear in the midth of any sequence* $\mathcal{S}_n^k$

By this corollary, we only have to check all the coprimes under the prime gap g. This condition could be a breakthrough when calculating big numbers unless k is a big prime. It is very likely k to have a small prime divisor q regardless big or small. Then we only have to check $\lceil g(1-\frac{1}{q}) \rceil$ numbers. More divisors k has, less numbers we have to check.

To combine all of these results above, we get the following algorithm for verification. In order to find the biggest prime number before kn in this study, we first generated a list of prime numbers with the maximum length $\mathbf{M}$ such that it does not raise any errors while searching. Then we can find prime numbers that are small enough with the time complexity $O(\log \mathbf{M})$. For the prime numbers that exceeds $\mathbf{M}$, we ran a primality test from $kn-1$ downwards to find the biggest prime number. Specifically, we used Miller-Rabin primality test [9] since the algorithm is very precise, also it is rather easier to implement on BigInteger.

Algorithm 1 Verifying k

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procedure verify(k)                                  ▷ Verify if k is all-to-one
    isall1 ← true
    define check                                     ▷ a data structure

    for i from 1 to g do
        n ← i
        clear orbit                                  ▷ a data structure
        while n ≠ 1 do
            if n is in check or (k, n) ≠ 1 then
                break
            else if n is in orbit then
                isall1 ← false
                break
            else
                put n in orbit
                put n in check
            end if

            p ← biggest prime before kn
            n ← kn − p
        end while

        if not isall1 then
            break
        end if

    end for
end procedure
```

2.3. Prime weighted power series

With a simple intuition, one can deduct that for any all-to-one k and any natural number n, it is possible to find a subsequence with length $l+1$ of $\mathcal{S}_n^k$ with first element n and last element 1. By the definition, we can easily show this equation holds.

$$k(k(\dots(kn - p_1) - p_2 \dots) - p_l = 1, \qquad p_i \textit{ is the biggest prime less than } k\mathcal{S}_n^k(i)$$

In other form,

$$n = \frac{1}{k^l} + \frac{p_l}{k^l} + \frac{p_{l-1}}{k^{1-1}} + \cdots + \frac{p_1}{k^1}.$$

Definition 5. If $n = \frac{1}{k^l} + \frac{p_l}{k^l} + \frac{p_{l-1}}{k^{1-1}} + \cdots + \frac{p_1}{k^1}$ for some $k \in \mathbb{N}$ and prime numbers $p_i, 1 \leq i \leq l$, we call it a **prime weighted** k**-power series of** n.

We have shown that a prime weighted k-power series of n exists for every natural number n and all-to-one k. But few questions arise from this expansion - Is the order of prime numerators unique? Or possibly is l subordinative to n and k? Moreover, is this series unique? Will changing

k to another all-to-one m give an integral sum again? Can a not-all-to-one k can have a prime weighted k-power series for every $n \in \mathbb{N}$? - we cannot answer these questions yet. But we may suggest this conjecture.

Conjecture 1. *A prime weighted k-power series of n is unique for all-to-one k and n, and it is the series derived from $\mathcal{S}_n^k$.*

3. Not-all-to-one k

3.1. Orbits

Theorem 8. *An orbit consists only odd numbers if k is even, and has even length if k is odd.*

PROOF. If k is even, $k \geq 2$. Then since every component n should satisfy $(k, n) = 1$ by Lemma 6. Since k is even, $(2, n) = 1$. Let k be odd and n be any component of the orbit. Since 3 is all-to-one, $k \geq 5$. Hence kn should be strictly bigger than 5 and thus the biggest prime p before kn cannot be 2. n and $kn - p$ cannot have the same parity. It implies that any orbit should have a length of even number.

For example, $k = 2572$ has an orbit of $3 \to 13 \to 9 \to 5 \to \cdots \to 11 \to 3$, which has only odd numbers as its component. Also, $k = 367352$ has an orbit of $3 \to 29 \to 9 \to 11 \to 39 \to \cdots \to 13 \to 7 \to 3$, and it has only odd numbers as its components.

On the other hands, $k = 2143$ has an orbit of $4 \to 9 \to 14 \to 13 \to \cdots \to 5 \to 4$, which has length of 8. Also, $k = 248917$ has an orbit of $5 \to 18 \to 25 \to 42 \to \cdots \to 39 \to 22 \to 5$, which has length of 20.

3.2. Distribution

By calculations, we have found every orbits with k less than 28314000. As we can see from the table, odd k's tend to have much more orbits. Regarding that there were 12994257 odd not-all-to-one k's and 10315630 even not-all-to-one k's, odd k's have likely to have more orbits. Besides, even k's seem to have more diverse orbit lengths. Though odd k's have much longer orbits, even k's' orbit lengths vary. Also, the more the length of the orbits gets longer, the less orbits there are. The number of the orbits even are strictly decreasing.

We cannot provide a thoroughful proof for the assymetrical distribution of orbits for odd and even k's, but intuitively it is pretty obvious that odd k's should have as twice as orbits than even k's. By Lemma 6, the components of an orbit should be coprimes to k. Hence even k's can have at most as half candidates for the orbit as odd k's.

Table 1: number of orbits

Length	Odd	Even	Length
2	19726872	8576763	2
4	4148803	2506421	3
6	1484385	1485506	4
8	569544	720840	5
10	208052	366303	6
12	69897	176677	7
14	21610	81581	8
16	6098	36033	9
18	1551	15234	10
20	386	6082	11
22	76	2270	12
24	16	795	13
26	3	271	14
-	-	85	15
-	-	25	16
-	-	9	17
-	-	1	18
-	26237293	13974896	-

On the other hands, this following table shows how many orbits the k's have. Very naturally, there tend to be less k's with more orbits. One interesting point is that there are more odd k's with 2 orbits rather than 1 orbit.

Table 2: number of orbits per k

# of orbits	Sum	Odd k	Even k
1	11854698	4675178	7179520
2	7409524	4752648	2656876
3	2929862	2492230	437632
4	875647	836290	39357
5	200321	198144	2177
6	34739	34672	67
7	4595	4593	2
8	449	449	-
9	47	47	-
10	5	5	-
-	23309887	12994257	10315630

4. Generalised $f(n) - p$ problem

Now we can expand this $kn - p$ problem using more broader approaches, such as generalised function actions on numbers.

Definition 6. We can define a sequence $\mathcal{S}_n^f = \{a_1, a_2, a_3 \dots\}$ for some function $f : \mathbb{N} \to \mathbb{N}$ and a natural number n, where $a_1 = n$ and $a_{i+1} = f(a_i) - p_i$ where p_i is the biggest prime number less than $f(a_i)$. If such p_i does not exist, $a_{i+1} = a_i$.

Definition 7. In similar sense to $kn - p$ problem, we will say that the function f has **all-to-one** property, alternatively f is **all-to-one** when $f : \mathbb{N} \to \mathbb{N}$ satisfies $\inf \mathcal{S}_n^f = 1$ for all $n \in \mathbb{N}$. Otherwise, we will say f is **not-all-to-one**.

Similar to the $kn - p$ problem, we want $\inf \mathcal{S}_n^f = 1$ for any n. With similar logic we used to solve the $kn - p$ problem, it is not so hard to show that numerous sublinear functions would be all-to-one since the sequence will be decreasing strictly for sufficiently big integers. For example, two functions $\lfloor \log_2 x \rfloor + 3$ or $\lfloor \sqrt{x} \rfloor + 7$ have all-to-one property. Moreover, we can show that some sublinear functions with specific form always have all-to-one property.

Theorem 9. *For any function in the form of $\lfloor \log_a x \rfloor + b$ is* all-to-one *regardless of the value of natural number $a \geq 3$ if $b = p + 1$ for some prime number p. In fact, b should be the next number of a prime number to have $\lfloor \log_a x \rfloor + b$ to be always* all-to-one *for all natural number a.*

PROOF. Let us think of a function $f = \lfloor \log_a x \rfloor + b$ where $a \geq 3$, and $b - 1$ is a prime number. (i.e. $b = p + 1$ for some prime number p.) Then if $n < a$, $f(n) = b = p + 1$ and hence $\inf \mathcal{S}_n^f = 1$. More specifically, $\mathcal{S}_n^f(2) = 1$. Now we know that if $n < a^{a-1}$, $\inf \mathcal{S}_n^f = 1$ since $f(n) = \lfloor \log_a n \rfloor + b < a - 1 + b = a + p$, and the biggest prime q smaller than $f(n)$ should be equal or bigger than p, $f(n) - q < a$. We can increase the upper bound of n with $\inf \mathcal{S}_n^f = 1$ because $a \geq 3$ gives the upper bound a^{a-1} is strictly bigger than the previous upper bound a. It is easy to see that for sufficiently big n, the sequence strictly decreases and therefore with finite steps of bound increasing we can show that f is all-to-one. To prove the second statement, simply having $a > b - p$ will make a cycle of $b - p \to b - p \to b - p \dots$, where p is the biggest prime number before b.

As we can see from the theorem, there are numerous numbers of sublinear all-to-one functions. But when it comes to superlinear functions, it is rather hard to see if a function is all-to-one. We can't even ensure if the sequence will be decreasing or not for big integers. It is pretty much similar to the notorious Collatz conjecture in this sense, which might imply that it will not be easy to show that a function is all-to-one. I will present some possible examples of a superlinear all-to-one function.

Conjecture 2. *$f(x) = x^2$, $g(x) = \lceil x \log x \rceil + 1$ are* all-to-one.

Oppermann's conjecture [10] suggests that there always exists a prime number between $n^2 - n$ and n^2 for natural number $n \geq 2$. This can imply that $\mathcal{S}_n^{x^2}$ always strictly decreasing, in other words, x^2 is all-to-one. If this conjecture holds, it is very likely that $\mathcal{S}_n^{x \log x}$ will decrease for sufficiently big integers and we can expect that these two functions, x^2 and $\lceil x \log x \rceil + 1$ are all-to-one. With numerical calculations, I could show that the sequences converge to 1 for small integers (smaller than 10,000,000). Though it is a very strong evidence that those two functions are all-to-one from the fact that maximal prime gap is sufficiently small, still there can be a very abnormal case of a cycle. In fact prime gap g_p grows less than the scale of $p_n^{0.525}$ [11], and it implies that if functions that are $o(n^{1/0.525}) \approx o(n^{1.9047})$ have all-to-one property with 'sufficiently small' integer domain, such functions will be all-to-one over whole natural number domain. This observation cannot assure that x^2 is all-to-one, but it is very likely $\lceil x \log x \rceil + 1$ to be all-to-one.

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